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THE MAXIMUM-FLOW, MINIMUM-CUT IN TRANSPORTATION PROBLEMS

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ABSTRACT

An important characteristic of a transport network is its capacity to carry the flow. In this work we develop a method of finding the maximum flow between source and target nodes of a network based on the "max-flow, min-cut" theorem in graph theory. The "max-flow, min-cut" theorem is used to find the set of links which limits the flow when we try to send maximum commodities from a given source to target node. Using this set of links, we can separate the network into two domains (source and target). In this paper, we have applied maximum–flow, minimum-cut using cut-set of the graph to the graph representing the transport network to direct the flow of commodity to its maximum capacity using the minimum number of edges. This result has got application in transportation problem to minimize the capacity of the cut and the cost of locating sensors in order to collect traffic data.

KEYWORDS: Cut-Set, Cut-Vertex, Capacity of Cut, Flow & Transport Network

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Original Article

1. INTRODUCTION

In graph theory, a flow network or transportation network is a directed graph where each edge has a capacity and each edge receives a flow. The amount of flow on an edge cannot exceed the capacity of the edge. In Operations Research, a directed graph is called a network, the vertices are called nodes and the edges are called arcs. A flow must satisfy the restriction that the amount of flow into a node equals the amount of flow out of it, except when it is a source, which has more outgoing flow, or sink, which has more incoming flow.

A network can be used to model traffic in a road system, fluids in pipes, currents in an electrical circuit, or anything similar in which something travels through a network of nodes. Networks have always been important in transportation and telecommunication also. However, they have become more important for all businesses today, especially because of the Internet. The Internet (originally conceived as a "network of networks") has connected virtually everything today. It has connected everybody, everything, everywhere into a network. Of course the internet has also changed how existing networks (e.g. transportation) behave.

In many applications, and especially in the planning of transportation and logistics of materials, one important problem is that of finding how much material can be shipped between two locations, using transportation links of limited capacity. This is also similar to the question that, what is the maximum flow that can be sent between any two nodes with given capacities on the edges. For example, traffic engineers may want to know the maximum flow rate of vehicles from the downtown car park to the freeway on-ramp because this will influence their decisions on whether to widen the roadways. Another example might be the maximum number of simultaneous telephone calls between two cities via the various lane-lines, satellites, and microwave towers

operated by a telephone company(Chinneck (2000)).

An infinite flow rate is impossible because the individual roads or telephone links have limited capacities to carry the flow. For example, there are usually multiple ways to drive between the downtown car park and the freeway on-ramp, or to route calls between two cities. Finding the maximum flow involves looking at all of the possible routes of flow between the two end points in question. When the system is mapped as a network, the arcs represent channels of flow with limited capacities. To find the maximum flow, assign flow to each arc in the network such that the total simultaneous flow between the two end point nodes is as large as possible.

In this paper, we provide a transportation problem of maximum flow and minimal cut of commodities through the network; i.e. the minimum number of edges in the network which has to be allowed and the maximum capacity of vehicles which can move through these edges.

2. DEFINITIONS

2.1 Cut-Set

In a connected graph *G*, a cut-set is a set of edges whose removal from *G* leaves *G* disconnected, provided removal of no proper subset of these edges disconnects *G*. A cut set always "cuts" a graph into two subgraphs. Therefore, a cut-set can also be defined as a minimal set of in a connected graph whose removal reduces the rank of the graph by one. Since removal of any edge from a tree breaks the tree into two parts, every edge of a tree is a cut-set.

2.2 Cut-Vertex

In a connected graph *G*, a cut-vertex is a set of vertices whose removal from *G* leaves *G* disconnected. If *v* is a cut vertex, then G-v have more components than *G*.

2.3 Edge Connectivity Of Graph

Each cut-set of a connected graph *G* contains of a certain number of edges. The number of edges in the smallest cut-set i.e. the cut-set with the fewest numbers of edges is defined as the edge connectivity of *G*. Equivalently, the edge connectivity of a connected graph can be defined as the minimal number of edges whose removal reduces the rank of the graph by one. To study the transportation problem, it has to be modelled mathematically by using a simple connected graph where the set of edges will represent the path between the set of vertices in the network.

A subset *F* of the set of edges of the graph *G* is a cut-set of the graph if the removal of F disconnects the graph (Baruah (2012, a), Chartrand (1977), Deo (2002)).

2.4 Weighted Graph

A weighted graph is a graph $G = (V, E)$ with a real number assigned on each of its edges. In a transportation problem the weight of an edge is often referred to as the "cost" of the edge. In applications, the weight may be a measure of the length of a route, the capacity of a line, the energy required to move between locations along a route, etc. Thus a graph with a real number $w(x, y)$ is called a weighted graph which serves as a structural model in transportation or traffic control problem(West (2005)).

The capacity of a flow is defined as the maximum hourly rate at which the traffic participants or vehicles can be expected to traverse the route during a given time period under prevailing roadway, traffic and control conditions (Baruah

(2012, b)). The time period normally used for expressing capacity is the hour.

2.5 Directed Graph Or Digraph

A directed graph *G* is a triple consisting of a vertex set *V*, an edge set *E* and a function assigning each edge an ordered pair of vertices. The first vertex of the order is the tail of the edge and the second is the head, together they are the endpoints(West (2005)).

3. CUT AND ITS CAPACITY

Ignoring the directions of edges in a transport network, let us consider a cut-set with respect to vertices *s* and *t*, that is, a cut-set which separates the sources from sink *t.* Such a set of edges in a transportation network is called a cut. The notation $(P,\bar{P})$ is used to denote a cut that partitions the vertices into two subsets P and $\bar{P}$, where P contains *s* and $\bar{P}$ contains *t*. The capacity of a cut denoted by $c(P,\bar{P})$ is defined to be the sum of the capacities of those edges directed from the vertices in set *P* to the vertices in $\bar{P}$; (Deo (2002)) that is,

$$\sum_{\substack{i\in P \\ j\in \bar{P}}} c_{ij} = c(P,\bar{P}) \tag{1}$$

Theorem3.1: Let *G = (V, E)* be a connected weighted graph where *V* is the set of vertices and *E* is the set of edges. The solution to the maximum flow problem is bounded above by the minimal cut capacity (Deo (2002)).

That is for a graph with cut-set or a cut that separates a pair of vertices, there exists atleast one cut $(P_{V_1,}P_{V_2})$ such that the flow from a vertex to another is equal to the capacity of the cut $(P_{V_1,}P_{V_2})$.

Theorem 3.2: Let *G* be a graph with source *s* and sink *t* with link capacities c_{ij}. Then the value of a max-flow from *s* to *t* is equal to the capacity of a min-cut between *s* and *t*(West (2005)).

Our main objective is to find a cut in the transport network, i.e. a minimal cut so that different commodities has to be allowed through these routes and the maximum amount of transportation of commodities through these routes. A cut in a transportation problem represents the right-of-way through which transportation is allowed from one point to another and whose removal will disconnect the flow. Also the edges of the cut represent the path in the network and hence it can be used for many purposes such as placing a sensor, as it provides necessary information about the flow of transport and it has got many applications in the real world.

4. FORMULATION AND SOLUTION OF THE PROBLEM

A transportation problem can be modelled by a transport or flow network, as it can be represented by a weighted connected graph in which the vertices are the different places and the edges are the routes through which transportation of vehicles and goods take place between these places.

In this problem, the weights are the real positive number, associated with each edge represents the capacity of the route, that is, the maximum amount of flow possible per unit of time.

In the graph represented in the figure below, we consider six vertices *s, a, b, c, d, t* and nine edges *sa,sb*, *ab*, *ac*, *bd*, *da*, *dc*, *ct*, *dt*, where the nodes represent various destinations and the edges represent the routes among the destinations. Also the capacity of each edge is represented by the weight assigned to it as shown in **Figure 1** below

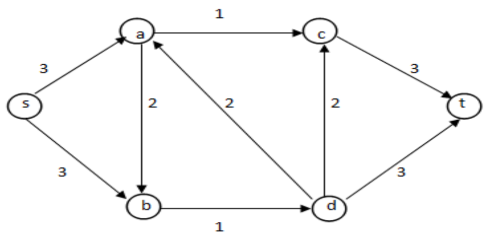

Figure 1: Network of Six Vertices

We are interested to find the maximum flow of commodities through the transport network together with the capacity. That is, we are interested to find the minimal cut in the network whose removal will disconnect the network with the maximum capacity. The term capacity of a transportation problem can be defined as the maximum amount of commodities that can pass a point on a lane or a roadway. When a road is carrying a traffic equal in volume to its capacity under ideal conditions, the operating conditions becomes poor, speeds drops down and delay and frequency of stops mounts up; which is not desired for a smooth flow of transport in a network.

Using equation (1), we can find the capacities of the cuts of the network in **Figure 1**. There are nine cuts which separate *s* from *t*, which are shown in **Table1**below.

Table 1: Cut-Sets of the Network in Figure 1

Vertex Set *P*	**Cut-Sets**	$c(P,\overline{P})$
{s}	{(s,a), (s,b)}	3+3=6 [**Figure 2**]
{s,a}	{(s,b), (a,b), (a,c), (d,a)}	3+2+1=6 [**Figure 3**]
{s,b}	{(s,a), (a,b), (b,d)}	3+1=4 [**Figure 4**]
{s,a,c}	{(s,b), (a,b), (d,a), (d,c), (c,t)}	3+2+3=8 [**Figure 5**]
{s,b,d}	{(s,a), (a,b), (d,a), (d,c), (d,t)}	3+2+2+3=10 [**Figure 6**]
{s,a,b}	{(a,c), (b,d), (d,a)}	1+1=2 [**Figure 7**]
{s,a,b,c}	{(b,d), (d,a),(d,c), (c,t)}	1+3=4 [**Figure 8**]
{s,a,b,d}	{(a,c), (d,a), (d,c),(d,t)}	1+2+3=6 [**Figure 9**]
{s,a,b,c,d}	{(c,t), (d,t)}	3+3=6 [**Figure 10**]

The figures of the cuts formed are given below.

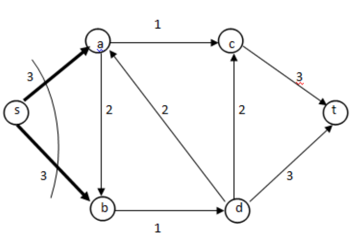

Figure 2: Cut-Vertex {s}

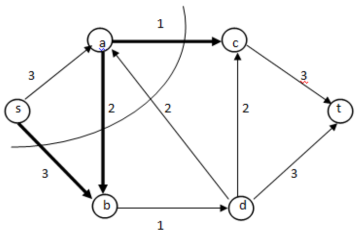

Figure 3: Cut-Vertex {s,a}

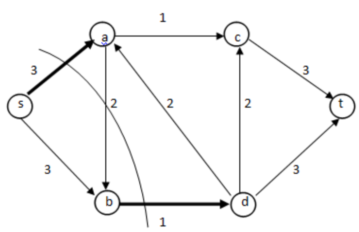

Figure 4: Cut-Vertex {s,b}

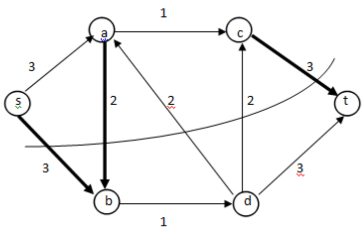

Figure 5: Cut-Vertex {s,a,c}

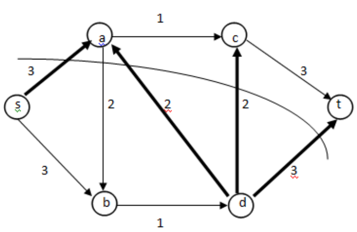

Figure 6: Cut-Vertex {s,b,d}

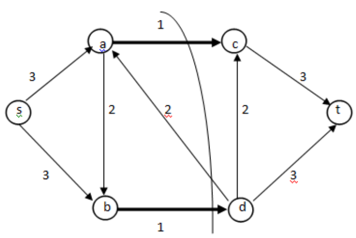

Figure 7:Cut-Vertex {s,a,b}

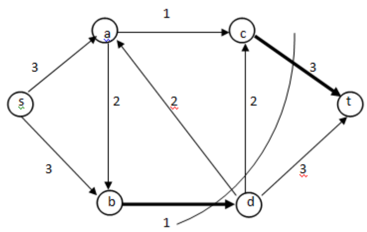

Figure 8: Cut-Vertex {s,a,b,c}

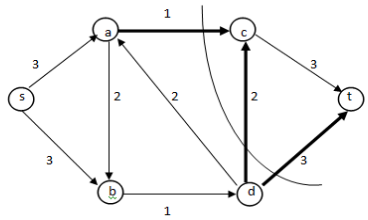

Figure 9: Cut-Vertex {s,a,b,d}

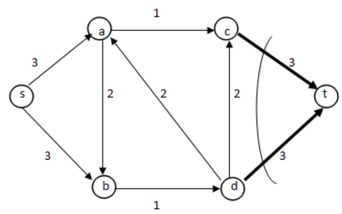

Figure 10:Cut-Vertex {s,a,b,c,d}

The cut with minimum capacity among these is the one in which $P=\{s,a,b\}$ and $\bar{P}=\{c,d,t\}$ (**Figure 7**). This cut represents that these edges has to be allowed for a smooth flow of transportation from one place to another. From **Theorem 3.1.,** the maximum flow possible from s to t in the network is therefore 2 units. This capacity of the cut is the minimum amount of commodities that can be allowed through the edges. Here cut concept is used to apply the theorem only, not to actually remove the edges. Since the capacity of the cut is minimum, there will be no effect on the flow of the network if we have to remove the cut edges. So the flow will be maximum.

Here we consider a network with directed edges. We can also consider the algorithm for undirected graphs. In that case we have to replace the edges of the undirected graph as two oppositely directed edges of the same weight.

5. APPLICATIONS

The problem of finding maximum flow capacity of networks has many applications in modern logistics management. Computer programs that can solve max-flow programs are used heavily by all automatic systems that calculate how shipments are transported by shipping companies. They may use planes, ships, trains, etc for transportation.

Another common use of max flow problems is in optimization of design. This includes design of piping systems for chemical and food-processing plants, water supply of a city, sewage systems, etc. One difference in the design of water supply systems is that usually there is one source (the reservoir), and many sinks (e.g. the water storage tank of each area, from where it is pumped to individual apartments). After completing the design of a particular supply network, we can then compute the maximum amount of water that can be supplied to any of the supply points. If this amount does not meet our required demand in that area, we will know that a higher capacity pipe should be designed

7. CONCLUSIONS

In the present transportation problem, we have used both cut-set of a graph and the weight to control the flow of transportation. In our problem, we have used the maximum flow and minimum cut, which is very useful in the damage condition of routs. If there is any damage situation like road blockage due to flood, then in this situation if the cut is minimum, then the flow should be maximum. This will help us in a smooth transportation of various commodities through the routs. Also this will help us in placing sensors and to minimize the cost of placing it. This concept can be generalized for a large number of destinations in a transportation system. The tools used here can be applied to formulate many other transportation problems having applications in the real world.

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